Local Unitary Equivalent Classes of Symmetric N-Qubit Mixed States

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Abstract. We develop a method of classifying LU equivalent classes of symmetric N-qubit mixed states based on multiaxial representation [1] of the density matrix. Bastin et.al [2] have defined two parameters, diversity degree and degeneracy configuration, to characterize symmetric N-qubit pure states using Majorana construction. This method can not be employed for symmetric mixed state classification. therefore we propose a more general method of LU classification for pure as well as mixed states based on the elegant multiaxial representation of the density matrix. In our scheme of classification, we introduce three parameters namely, diversity degree, degeneracy configuration, rank of the spherical tensor parameters which characterize the density matrix. The power of our method is demonstrated using several well known examples of symmetric two, three qubit pure states and two qubit mixed states. A recipe to identify the most general symmetric N-qubit pure separable state is also given.

Keywords: Local Unitary equivalence, Multiaxial Representation

1 Introduction

Local unitary (LU) equivalence of multipartite pure states has received a lot of attention recently [3-6]. ρ and ρ' are said to be LU equivalent if $\rho' = U\rho U^\dagger$ where $U \in SU(2)^{\times N}$. Local Operation and Classical Communication (LOCC) equivalent classes are defined such that all quantum states within the same class can be transformed to each other by LU transformation. [3]. It is well known that the states belonging to the same LU equivalent class can be used for similar quantum information processing tasks as they possess the same amount of entanglement. One way of classifying these states is by evaluating the LU invariants. Well known algebraic methods for generation of invariants already exist in literature [7-10]. As the number of subsystems increases, the problem of identifying and interpreting the independent invariants rapidly becomes very complicated. However, LU invariants associated with the symmetric states, which are experimentally viable and mathematically elegant, are easier to handle as the dimensionality of the Hilbert space involved is much less. Because of the permutational symmetry involved in the symmetric state, ρ and ρ' are said to belong to the same LU equivalent class if $\rho' = R\otimes R\otimes R...R\,\rho\, R^{-1}\otimes R^{-1}...R^{-1}$ where R represents the rotation operator on a qubit [11]. Different LU equivalent classes of up to 5 pure qubit states and for certain mixed states have been determined by introducing the standard form for multipartite states [4, 5]. LU invariants of the most general symmetric mixed systems have been constructed using the elegant multiaxial representation of the symmetric states [12]. But entanglement classification of mixed state under local transformation poses a difficult problem as the definition of mixed state entanglement itself is poorly understood. However, an operational entanglement classification of symmetric mixed state for an arbitrary number of qubits under Stochastic Local Operation and Classical Communication (SLOCC) has been introduced by Bastin et. al[13]. In this paper we propose a scheme for classifying the most general symmetric N-qubit mixed states under LU transformation based on the multi-axial representation of Ramachandran and Ravishankar [1].

Set of N-qubit pure states that remain unchanged by permutations of individual particles are called symmetric states. Symmetric states offer elegant mathematical analysis as the dimension of the Hilbert space reduces drastically from 2^N to $(N+1)$, when N-qubits respect exchange symmetry. Such a Hilbert space is considered to be spanned by the eigen states $\{|j,m\rangle; -j \le m \le +j\}$ of angular momentum operators J^2 and J_z , where $j = \frac{N}{2}$. Fortunately, a large number of experimentally relevant states [14] possesses symmetry under particle exchange and this property allows us to significantly reduce the computational complexity.

2 Multiaxial Representation of Density Matrix

The standard expression for the most general spin-j density matrix or symmetric N-qubit density matrix in terms of Fano statistical tensor parameters $t^k_q\,'s$ is given by [15]

$$\rho(\vec{J}) = \frac{Tr(\rho)}{(2j+1)} \sum_{k=0}^{2j} \sum_{q=-k}^{+k} t^k_q\, \tau^{k^\dagger}_q(\vec{J}) \ , \qquad (1)$$

where $\vec{J}$ is the angular momentum operator with components $\vec{J}_x, \vec{J}_y, \vec{J}_z$ and $\tau^k_q\,'s$ (with $\tau^0_0 = I$,the identity operator) are irreducible tensor operators of rank k in the $2j+1$ dimensional spin space with projection q along the axis of quantization in the real 3-dimensional space. The spherical tensor parameters $t^k_q\,'s$ are the average expectation values $t^k_q = Tr(\rho\,\tau^k_q)$. Since ρ is Hermitian and $\tau^{k^\dagger}_q = (-1)^q \tau^k_{-q}$, complex conjugate of $t^k_q\,'s$ satisfy the condition

$$t^{k^*}_q = (-1)^q\, t^k_{-q} \ . \qquad (2)$$

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It has been shown by Ramachandran and Ravishankar[1] that any spherical tensor t_q^k of rank k can be represented geometrically by a set of k vectors $\{\hat{Q}(\theta_i, \varphi_i)\}$; $i = 1, ..., k$ on the surface of a sphere of radius r_k. As the state of spin- j assembly is characterized by $2j$ spherical tensors, the state can be represented geometrically by a set of $2j$ spheres of radii $r_1, r_2, ..., r_{2j}$ having $1, 2, ..., 2j$ vectors specified on its surface, respectively. Thus, the spin-j system is in general characterized by $j(2j+1)$ axes and $2j$ scalars. Since scalar product between any two vectors $\hat{Q}(\theta_i, \varphi_i)$ and $\hat{Q}(\theta_j, \varphi_j)$ is an invariant under rotation, a spin-j or symmetric N-qubit density matrix is characterised by $C_2^{j(2j+1)} + 2j$ invariants [12] where $C_2^{j(2j+1)}$ denotes the binomial coefficient.

3 LU classification

A symmetric N-qubit pure state is given by

$$|\psi\rangle = \mathcal{N} \sum_{1 \leq i_1 \neq ... \neq i_N \leq N} |\epsilon_{i_1} ... \epsilon_{i_N}\rangle \qquad (3)$$

where $\mathcal{N}$ is a normalization factor and the $|\epsilon_i\rangle\,'s$ are single qubit states $|\epsilon_i\rangle = \alpha_i|1\rangle + \beta_i|0\rangle$ with $|\alpha_i|^2 + |\beta_i|^2 = 1$. According to Bastin et. al, [2], one needs two parameters namely diversity degree d and degeneracy configuration $\mathcal{D}_{\{n_i\}}$ for state classification based on Majorana representation. For example, a symmetric N-qubit state with all ϵ_i identical has a degeneracy configuration $\mathcal{D}_N$ and a diversity degree d of 1. If all but one ϵ_i are identical, we get the configuration $\mathcal{D}_{N-1,1}$ and $d = 2$. If all but two ϵ_i are identical, we get the configuration $\mathcal{D}_{N-2,2}(d = 2)$ or $\mathcal{D}_{N-2,1,1}(d = 3)$, depending on whether the two remaining ones are identical or not, respectively. This classification is applicable for symmetric pure states only. Here we propose a scheme for classification of both pure as well as mixed symmetric states based on multiaxial representation. In our scheme, every t^k is characterised by three parameters namely, diversity degree, degeneracy configuration and rank of the spherical tensor t^k. For example, in the case of a symmetric two qubit mixed state, the density matrix is characterized by t_q^1 and t_q^2. In the case of t^1, there is only one axis. Thus $t^1 \in \mathcal{D}_1^1$. In the case of t^2, there are two axes in general. If the two axes are identical then $t^2 \in \mathcal{D}_2^2$ and if the axes are not collinear then $t^2 \in \mathcal{D}_{1,1}^2$. Thus the two qubit system belongs to one of the following classes.

(i) $\{\mathcal{D}_1^1\} \longrightarrow \rho$ is pure vector polarized.
(ii) $\{\mathcal{D}_2^2\} \longrightarrow \rho$ is pure tensor polarized.
(iii) $\{\mathcal{D}_1^1, \mathcal{D}_2^2\} \longrightarrow \rho$ is oriented.
(iv) $\{\mathcal{D}_1^1, \mathcal{D}_{1,1}^2\} \longrightarrow \rho$ is triaxial and non-oriented.

Similarly, a symmetric three qubit mixed system is characterized by t^1, t^2 and t^3. Thus the spherical tensor parameters of the spin-3/2 or symmetric three qubit density matrix belong to the following configurations.

(i) $t^1 \in \mathcal{D}_1^1$.
(ii) $t^2 \in \mathcal{D}_2^2$ or $t^2 \in \mathcal{D}_{1,1}^2$.
(iii) $t^3 \in \mathcal{D}_3^3$ or $t^3 \in \mathcal{D}_{2,1}^3$ or $t^3 \in \mathcal{D}_{1,1,1}^3$.

Observe that two density matrices ρ and ρ' are said to be LU equivalent if they posses the same set of invariants. Thus, in our scheme of classification it is obvious that every spherical tensor of a given rank k characterising ρ and ρ' should have the same degeneracy configuration and diversity degree in order to posses the same number of invariants. For example, a symmetric two qubit density matrix having the degeneracy configuration $\{\mathcal{D}_1^1, \mathcal{D}_2^2\}$ is not LU equivalent to another symmetric two qubit density matrix with degeneracy configuration $\{\mathcal{D}_1^1, \mathcal{D}_{1,1}^2\}$ since they posses different set of invariants.

Employing the above classification scheme, a recipe for identifying N-qubit pure separable state is discussed in detail. Some well known examples of symmetric two and three qubit pure states are also investigated. Classification of uniaxial, biaxial and triaxial two qubit mixed states which can be produced in the laboratory is illustrated.

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